

## Unusual Liénard-type nonlinear oscillator

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A Liénard type nonlinear oscillator of the form  $\ddot{x} + kx\dot{x} + (k^2/9)x^3 + \lambda_1 x = 0$ , which may also be considered as a generalized Emden-type equation, is shown to possess unusual nonlinear dynamical properties. It is shown to admit explicit nonisolated periodic orbits of conservative Hamiltonian type for  $\lambda_1 > 0$ . These periodic orbits exhibit the unexpected property that the frequency of oscillations is completely independent of amplitude and continues to remain as that of the linear harmonic oscillator. This is completely contrary to the standard characteristic property of nonlinear oscillators. Interestingly, the system though appears deceptively a dissipative type for  $\lambda_1 \leq 0$  does admit a conserved Hamiltonian description, where the characteristic decay time is also independent of the amplitude. The results also show that the criterion for conservative Hamiltonian system in terms of divergence of flow function needs to be generalized.

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### I. INTRODUCTION

Nonlinear oscillator systems are ubiquitous and they model numerous physical phenomena ranging from atmospheric physics, condensed matter, nonlinear optics to electronics, plasma physics, biophysics, evolutionary biology, etc. [1–5]. One of the most important characteristics of nonlinear oscillations is the amplitude or initial condition dependence of frequency for nonisolated periodic orbits [1,3,5]. For example, for the cubic anharmonic oscillator

$$\ddot{x} + \omega_0^2 x + \beta x^3 = 0 \quad (\omega_0^2, \beta > 0), \quad (1)$$

where the overdot denotes differentiation with respect to  $t$ , the general (periodic) solution is  $x(t) = A \operatorname{cn}(\Omega t + \delta)$ , where  $A$  and  $\delta$  are arbitrary constants with the modulus squared of the Jacobian elliptic function  $m^2 = \beta A^2 / 2(\omega_0^2 + \beta A^2)$  and frequency  $\Omega = \sqrt{\omega_0^2 + \beta A^2}$ . In the case of limit cycle oscillations the dependence of the initial condition is manifested in the form of suitable transient time to reach the asymptotic state. For chaotic oscillations, of course, there is the sensitive dependence on initial conditions. In this paper we identify a physically interesting and simple nonlinear oscillator of the Liénard type, which is also a generalized Emden-type equation, that admits for a particular sign of the control parameter nonisolated conservative periodic oscillations, exhibiting the remarkable fact that the frequency of oscillation is completely independent of amplitude and remains the same as that of the linear harmonic oscillator, thereby showing that the amplitude dependence of frequency is not necessarily a fundamental property of nonlinear dynamical phenomena. We also show that rewriting the underlying equation of motion as a system of two first order coupled nonlinear differential equations the basic criterion for conservative system in terms of the divergence of the flow function has to be generalized.

We consider a nonlinear oscillator system of the form

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x = 0, \quad (2)$$

which is of the Liénard type  $\ddot{x} + f(x)\dot{x} + g(x) = 0$ , where  $f(x) = kx$  and  $g(x) = (k^2/9)x^3 + \lambda_1 x$  in the present case. Equa-

tion (2) can also be interpreted as the cubic anharmonic oscillator defined by Eq. (1) (with  $\omega_0^2 = \lambda_1$  and  $\beta = k^2/9$ ), but acted upon on by a strong damping type nonlinear force  $kx\dot{x}$ . This type of equation is also well studied in the literature for almost two decades as a generalized form of Emden-type equation occurring in the study of equilibrium configurations of a spherical cloud acting under the mutual attraction of its molecules and subject to the laws of thermodynamics [6]. Equation (2) is a special case of a general second order nonlinear differential equation possessing eight Lie point symmetries [7] and that it is linearizable. In particular the special case  $\lambda_1 = 0$  has been well analyzed by Mahomed and Leach [7] and explicit forms of the eight symmetry generators satisfying  $sl(3, \mathbb{R})$  algebra have been well documented. The latter case ( $\lambda_1 = 0$ ) has the exact general solution

$$x(t) = \frac{t + I_1}{\frac{k}{6}t^2 + \frac{I_1 k}{3}t + I_2}, \quad (3)$$

where  $I_1$  and  $I_2$  are the two integrals of motion with the explicit forms

$$I_1 = -t + \frac{x}{\frac{k}{3}x^2 + \dot{x}}, \quad (4a)$$

$$I_2 = \frac{k}{6}t^2 + \frac{1 - \frac{k}{3}tx}{\frac{k}{3}x^2 + \dot{x}}. \quad (4b)$$

Obviously for  $\lambda_1 = 0$  case the initial value problem of Eq. (2) appears to be a dissipative nonlinear system.

In the following we show that the nonlinear oscillator Eq. (2) possesses several unusual and interesting features, which are contrary to the standard characteristic features associated with usual nonlinear oscillators. For example, in the case  $\lambda_1 > 0$  we prove that the system (2) possesses oscillatory motion whose frequency is completely independent of the

amplitude. Thus we prove that the standard notion of amplitude-dependent frequency of oscillations is not a necessary condition for nonlinear oscillators. Moreover, we also point out that the system (2) admits a Lagrangian formulation and from which a conservative Hamiltonian can be obtained. Further, we also find that the standard test (divergence of the flow function) to identify whether the given system is conservative or not fails here. To overcome this situation we propose that the necessary condition for a system to be a conserved Hamiltonian is that the time average of the divergence of the flow function should vanish instead of the flow function itself vanishing. More interestingly, we also find that the system (2) when  $\lambda_1 \leq 0$  also admits a Lagrangian as well as a conserved Hamiltonian description similar to the case  $\lambda_1 > 0$ , even though the general solution shows a dissipative/damping/frontlike aperiodic structure, depending on the initial condition. The divergence of the flow function for the case  $\lambda_1 \leq 0$  can be negative for all times in spite of the conserved Hamiltonian nature. Again to redeem the situation, one requires the time average of the divergence of the flow function to be vanishing. Further, we point out that the special features of the system can be traced to the existence of certain linearizing and canonical and nonlocal transformations, which points out to generalization of the results in different directions. Finally, we also note that the system (2) is a typical example of a reversible dynamical system, well studied in the literature [8–19], and bring out the importance of reversible symmetries in relation to the above explicit dynamical behaviours associated with Eq. (2).

The plan of the paper is as follows. In the following section, we utilize the so-called modified Prolle-Singer procedure and derive two functionally independent integrals of motion for the Eq. (2). In Sec. III, we show that for the case  $\lambda_1 > 0$  the system (2) admits periodic solutions and construct the associated Lagrangian as well as Hamiltonian for this system. We discuss the unusual features associated with the system in Sec. IV and point out the amplitude independence of the frequency of oscillations as well as the nonzero value of the divergence of the flow function in spite of its conservative nature. We analyse the nature of solutions in the regimes  $\lambda_1 = 0$  and  $\lambda_1 < 0$  in Sec. V and point out that in spite of the decaying/damping nature or frontlike aperiodic nature of the solutions, the system admits Lagrangian and Hamiltonian descriptions. In Sec. VI we have indicated some interesting connections between our system (2) and reversible dynamical systems. In Sec. VII, we linearize the Eq. (2) through different kinds of transformations (invertible point, nonlocal and canonical transformation) to trace the unusual features of the system. In Sec. VIII, we generalize Eq. (2) and compare the dynamics with certain other interesting nonlinear oscillators. Finally, we present our conclusions in Sec. IX.

## II. EXTENDED PRELLE-SINGER PROCEDURE

Now let us consider the general case  $\lambda_1 \neq 0$ . One can proceed to solve the equation explicitly by the so called modified Prolle-Singer (PS) method [20,21] which identifies the integrals of motion and explicit solution,

if they exist. One can also use other methods as well; however, we find that the PS method is quite convenient [21] to obtain both the integrals of motion and solution explicitly. Assuming the existence of an integral  $I = I(t, x, \dot{x})$  for Eq. (2) and rewriting the latter as  $\ddot{x} = P(x, \dot{x})$ , where  $P = -kx\dot{x} - (k^2/9)x^3 - \lambda_1 x$ , so that  $Pdt - d\dot{x} = 0$ , and introducing a null term  $S(t, x, \dot{x})\dot{x}dt - S(t, x, \dot{x})dx$ , we find that on the solutions,  $(P + S\dot{x})dt - Sdx - d\dot{x} = 0$ . Since  $I$  is a constant of motion, it follows that

$$dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x} = 0, \quad (5)$$

so that

$$dI = R(P + S\dot{x})dt - RSdx - Rd\dot{x} = 0, \quad (6)$$

where  $R$  is an integrating factor. Comparing Eqs. (5) with (6) we have, on the solutions, the relations

$$I_t = R(P + \dot{x}S), \quad I_x = -RS, \quad I_{\dot{x}} = -R.$$

Then the compatibility conditions,  $I_{tx} = I_{xt}$ ,  $I_{t\dot{x}} = I_{\dot{x}t}$ ,  $I_{x\dot{x}} = I_{\dot{x}x}$ , require that

$$\begin{aligned} D[S] &= -P_x + SP_{\dot{x}} + S^2, \\ D[R] &= -R(S + P_{\dot{x}}), \\ R_{\dot{x}} &= R_x S + RS_{\dot{x}}, \end{aligned} \quad (7)$$

where

$$D = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + P \frac{\partial}{\partial \dot{x}}.$$

Solving Eq. (7) systematically for  $S$  and  $R$ , one can write down the form of the integral of motion

$$I = r_1 - r_2 - \int \left[ R + \frac{d}{d\dot{x}}(r_1 - r_2) \right] d\dot{x}, \quad (8)$$

where

$$r_1 = \int R(P + \dot{x}S)dt, \quad r_2 = \int \left( RS + \frac{d}{dx}r_1 \right) dx,$$

for the given form  $P = -kx\dot{x} - (k^2/9)x^3 - \lambda_1 x$ . We find two independent sets of compatible solutions of Eq. (7), namely,

$$S_1 = \frac{-\dot{x} + \frac{k}{3}x^2}{x}, \quad R_1 = \frac{xe^{-2\sqrt{-\lambda_1}t}}{\left(\dot{x} + \frac{k}{3}x^2 - \sqrt{\lambda_1}x\right)^2}, \quad (9a)$$

$$S_2 = \frac{\left(\frac{k}{3}x + \sqrt{-\lambda_1}\right)^2 - \frac{k}{3}\dot{x}}{\frac{k}{3}x + \sqrt{-\lambda_1}}, \quad R_2 = \frac{\left(\frac{k}{3}x + \sqrt{-\lambda_1}\right)e^{\sqrt{-\lambda_1}t}}{\left(\dot{x} + \frac{k}{3}x^2 + \sqrt{-\lambda_1}x\right)^2}. \quad (9b)$$

Consequently, substituting  $R$  and  $S$  in Eq. (8), we find the two (time dependent) integrals of motions for Eq. (2) as

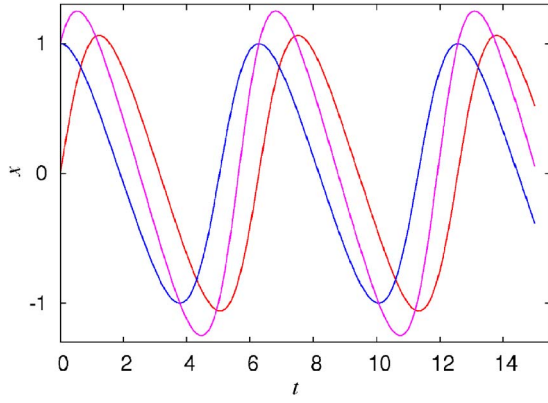


FIG. 1. (Color online) Harmonic periodic solutions of the nonlinear Liénard-type oscillator for different initial conditions [Eq. (2)] with frequency same as that of the linear harmonic oscillator.

$$I_1 = e^{-2\sqrt{-\lambda_1}t} \left( \frac{\dot{x} + \frac{k}{3}x^2 + \sqrt{-\lambda_1}x}{\dot{x} + \frac{k}{3}x^2 - \sqrt{-\lambda_1}x} \right), \quad (10a)$$

$$I_2 = -\frac{6}{k} e^{\sqrt{-\lambda_1}t} \left( \frac{\lambda_1 + \frac{k}{3}\dot{x} + \frac{k^2}{9}x^2}{\dot{x} + \frac{k}{3}x^2 + \sqrt{-\lambda_1}x} \right). \quad (10b)$$

### III. PERIODIC SOLUTIONS, LAGRANGIAN AND HAMILTONIAN DESCRIPTION FOR THE CASE $\lambda_1 > 0$

As we mentioned in the introduction Eq. (2) admits two different kinds of dynamics depending on the sign of the parameter  $\lambda_1$ . In this section we discuss the case  $\lambda_1 > 0$ .

#### A. Periodic solutions

Now we note that for  $\lambda_1 > 0$  both the integrals  $I_1$  and  $I_2$  become complex. To identify two real integrals, we consider the combinations

$$\mathfrak{J}_1 = \frac{4}{k^2 I_1 I_2^2} = \frac{(3\dot{x} + kx^2)^2 + 9\omega^2 x^2}{(3k\dot{x} + k^2 x^2 + 9\omega^2)^2}, \quad (11)$$

where  $\omega = \sqrt{\lambda_1}$  and

$$\mathfrak{J}_2 = -\frac{2e^{i\delta}}{k|I_1 I_2|} = e^{i(\omega t + \delta)} \left( \frac{3\dot{x} + kx^2 - 3i\omega x}{3k\dot{x} + k^2 x^2 + 9\omega^2} \right), \quad (12)$$

so that  $\mathfrak{J}_1$  and  $|\mathfrak{J}_2|$  can be taken as the two real integrals of Eq. (2) for  $\lambda_1 > 0$ . In Eq. (12),  $\delta$  is a phase constant.

Thus for the case  $\lambda_1 > 0$ , the integrals (11) and (12) lead to the explicit sinusoidal periodic solution (Fig. 1),

$$x(t) = \frac{A \sin(\omega t + \delta)}{1 - \left(\frac{k}{3\omega}\right) A \cos(\omega t + \delta)}, \quad 0 \leq A < \frac{3\omega}{k}, \quad \omega = \sqrt{\lambda_1}, \quad (13)$$

where  $A = 3\omega\sqrt{\mathfrak{J}_1}$  and  $\delta$  is an arbitrary constant. The form of the first integral (11) also establishes that the system is of conservative type. Note that in Eq. (13)

$$x_{\max} = B = \frac{A}{\left(1 - \frac{k^2}{9\omega^2} A^2\right)^{1/2}}$$

and

$$x_{\min} = -\frac{A}{\left(1 - \frac{k^2}{9\omega^2} A^2\right)^{1/2}},$$

so that

$$A = \pm \frac{B}{\left(1 + \frac{k^2}{9\omega^2} B^2\right)^{1/2}}$$

and  $B$  may be called as the amplitude of oscillations in the present case. Also for  $A \geq 3\omega/k$ , the solution (13) becomes singular at finite times.

#### B. Lagrangian and Hamiltonian description

The form of the time independent first integral  $\mathfrak{J}_1$  as given by Eq. (11) suggests that one can indeed give a Lagrangian and so a Hamiltonian description for Eq. (2) when  $\lambda_1 > 0$ . Indeed we identify a suitable Lagrangian for the system (2) as

$$L = \frac{27\lambda_1^3}{2k^2} \left( \frac{1}{k\dot{x} + \frac{k^2}{3}x^2 + 3\lambda_1} \right) + \frac{3\lambda_1}{2k} \dot{x} - \frac{9\lambda_1^2}{2k^2}. \quad (14)$$

Then the canonically conjugate momentum can be defined as

$$p = \frac{\partial L}{\partial \dot{x}} = -\frac{27\lambda_1^3}{2k} \left( \frac{1}{\left(k\dot{x} + \frac{k^2}{3}x^2 + 3\lambda_1\right)^2} \right) + \frac{3\lambda_1}{2k}. \quad (15)$$

Consequently, one can obtain the Hamiltonian associated with Eq. (2) as

$$H = \frac{9\lambda_1^2}{2} \left( \frac{\left(\dot{x} + \frac{k}{3}x^2\right)^2 + \lambda_1 x^2}{\left(k\dot{x} + \frac{k^2}{3}x^2 + 3\lambda_1\right)^2} \right) \equiv \frac{9\lambda_1^2 \mathfrak{J}_1}{2} \quad (16a)$$

or

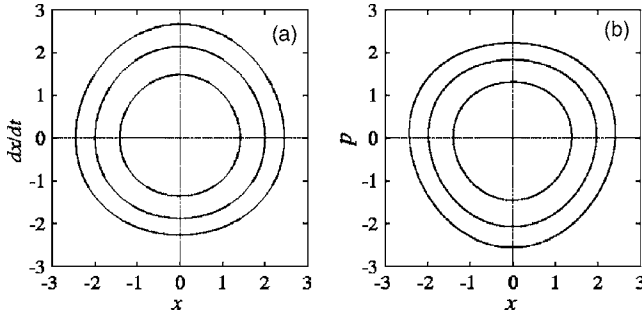


FIG. 2. Phase trajectories corresponding to Eq. (2) for  $|x| < 3\omega/k$ : (a)  $(x-\dot{x})$  plane, (b)  $(x-p)$  plane.

$$H = \frac{9\lambda_1^2}{2k^2} \left[ 2 - 2 \left( 1 - \frac{2kp}{3\lambda_1} \right)^{1/2} + \frac{k^2x^2}{9\lambda_1} - \frac{2kp}{3\lambda_1} - \frac{2k^3x^2p}{27\lambda_1^2} \right]. \quad (16b)$$

Note that in the above expressions  $L$ ,  $p$  and  $H$ , we have kept certain prefactors and constant terms in order to have the correct limits for  $k \rightarrow 0$ , namely, the harmonic oscillator limit. The nature of the trajectories in the  $(x-\dot{x})$  and  $(x-p)$  planes are shown in Fig. 2. They form closed concentric curves in the region  $|x| < 3\omega/k$ .

Finally we may point out that because of the form of the second and third terms in the right-hand side of Eq. (14) one may define a modified Lagrangian

$$\bar{L} = \frac{1}{k\dot{x} + \frac{k^2}{3}x^2 + 3\lambda_1}, \quad (17a)$$

with the conjugate momentum

$$\bar{p} = - \frac{k}{\left( k\dot{x} + \frac{k^2}{3}x^2 + 3\lambda_1 \right)^2} \quad (17b)$$

and the associated Hamiltonian

$$\bar{H} = \frac{2k\dot{x} + \frac{k^2}{3}x^2 + 3\lambda_1}{\left( k\dot{x} + \frac{k^2}{3}x^2 + 3\lambda_1 \right)^2} \quad (17c)$$

or

$$H = \left( \frac{k}{3}x^2 + \frac{3\lambda_1}{k} \right) p - 2 \sqrt{\frac{-p}{k}}, \quad (17d)$$

which again correspond to the equation of motion (2), provided  $k \neq 0$ . However, in the limit  $k \rightarrow 0$ , in order to get the correct harmonic oscillator it is preferable to have the forms (14)–(16). On the other hand, the later forms (17) have the advantage that they give interesting  $\lambda_1 = 0$  limit, which we discuss in the next section. Consequently, we have chosen to present both the forms here.

## IV. UNUSUAL FEATURES OF EQ. (2) WITH $\lambda_1 > 0$

### A. Amplitude-independent frequency of oscillations

The remarkable features of the general solution (13) are that the nonlinear system (2) admits harmonic periodic solutions (13) and that the frequency of these periodic oscillations,  $\nu = \omega/2\pi = \sqrt{\lambda_1}/2\pi$ , is completely independent of the amplitude or initial condition, a feature which is generally quite uncommon for nonlinear oscillators of conservative type. The form of the solution is shown in Fig. 1. It may be noted that in the limiting case of vanishing nonlinearity,  $k=0$ , one recovers the solution of the linear harmonic oscillator as it should be from Eq. (13) and there is no change in the angular frequency  $\omega = \sqrt{\lambda_1}$  even when  $k \neq 0$ , as can be seen from Eq. (13). However, it may be noted that in the limit  $\lambda_1$  or  $\omega \rightarrow 0$ ,  $A \rightarrow 0$  and so  $x \rightarrow 0$ , the trivial solution to Eq. (2) and not to the aperiodic bounded solution (3). Thus at  $\lambda_1 = 0$ , a bifurcation occurs.

### B. The nature of flow function

Another interesting feature of Eq. (2) is that when written as a system of two first order equations, it takes the form

$$\dot{x} = y \equiv f_1(x, y), \quad (18a)$$

$$\dot{y} = -kxy - \frac{k^2}{9}x^3 - \lambda_1x \equiv f_2(x, y). \quad (18b)$$

For  $\lambda_1 > 0$ , it has one equilibrium point (0,0) which is of centre type compatible with the occurrence of nonisolated periodic orbits. Interestingly the conventional criterion for conservative systems (see, for example, Refs. [22–25]), namely, the value of the divergence of the flow function  $\Lambda = \partial f_1/\partial x + \partial f_2/\partial y$  for Eq. (18) vanish, fails here as  $\Lambda = -kx$ . Only the average of  $\Lambda$ , namely,

$$\bar{\Lambda} = \frac{1}{T} \int_0^T \Lambda dt = - \frac{k\omega}{2\pi} \int_0^{2\pi/\omega} x dt, \quad (19)$$

on actual evaluation using the solution (13), vanishes. Such a generalized criterion that the average of the flow function vanishes for a conservative system seems to be a necessary condition as the present example shows (see also the last paragraph of Sec. V below and [9]).

## V. THE APERIODIC CASES $\lambda_1 \leq 0$

In the previous section, we discussed the dynamics of Eq. (2) for the case  $\lambda_1 > 0$ . In the following we explore the dynamics of the system (2) with  $\lambda_1 \leq 0$ . For  $\lambda_1 < 0$ , both the time-dependent integrals (10) are real, from which the explicit solution can be obtained straightforwardly as

$$x(t) = \left( \frac{3\sqrt{|\lambda_1|}(I_1 e^{2\sqrt{|\lambda_1|}t} - 1)}{kI_1 I_2 e^{\sqrt{|\lambda_1|}t} + k(1 + I_1 e^{2\sqrt{|\lambda_1|}t})} \right), \quad (20)$$

where  $I_1$  and  $I_2$  are constants. Solution (20) clearly shows the dissipative/damping/aperiodic nature of the system for  $\lambda_1 \leq 0$  in Eq. (2).

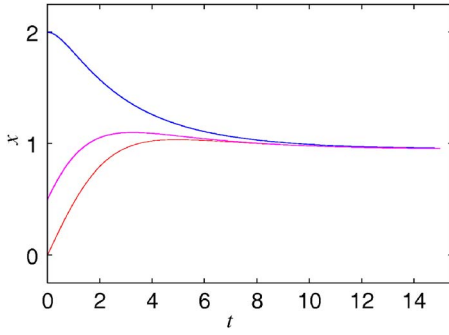


FIG. 3. (Color online) Solution plot for case  $\lambda_1 < 0$  for different initial conditions.

We now note that system  $\lambda_1 \leq 0$  still admits the Lagrangian and Hamiltonian descriptions. In fact the Lagrangian  $L$  and the Hamiltonian  $H$  for the case  $\lambda_1 > 0$ , namely, the forms (14) and (16), respectively, are valid here also. The only difference here is that one has to replace  $\lambda_1$  with  $-\lambda_1$  in the respective places in Eqs. (14) and (16) for the present case. On the other hand the forms (17) are valid for  $\lambda_1 > 0$  as well as  $\lambda_1 \leq 0$ , though not valid in the limiting harmonic oscillator case  $k \rightarrow 0$ .

The major difference in the dynamics comes from the nature of the solution. In the present case it admits decaying type or aperiodic (frontlike) solutions only, see Fig. 3, where the time of decay or approach to asymptotic value is independent of the amplitude/initial value, which is once again an unusual feature for a nonlinear dynamical system.

We mention here that in the case  $|\lambda_1| < 0$  when the nonlinearity is absent in Eq. (2), it becomes a repulsive harmonic oscillator, namely,  $\ddot{x} - |\lambda_1|x = 0$ , which obviously admits an unbounded solution of the form,  $x(t) \propto e^{\pm|\lambda_1|t}$  as  $t \rightarrow \pm\infty$ . However, the presence of the nonlinearity terms,  $kx\dot{x} + (k^2/9)x^3$ , makes the solution (20) to be a bounded and frontlike one with asymptotic values  $x(t) \rightarrow \pm 3\sqrt{|\lambda_1|}/k$  as  $t \rightarrow \pm\infty$ .

Further, from the Fig. 3, one can see that for any initial condition the frontlike solution asymptotically approaches the value  $3\sqrt{|\lambda_1|}/k$  as  $t \rightarrow \infty$ , which is a stable node (see also Sec. VI below), as may be verified from the explicit solution (20) itself. Thus the frontlike structure (20) is a linearly stable entity.

The nontrivial general solution for the limiting case  $\lambda_1 = 0$  is given in Eq. (3). Interestingly in this case also one can deduce a Lagrangian and the associated Hamiltonian, which [from Eqs. (17)] turn out to be

$$\hat{L} = \left( \frac{1}{k\dot{x} + \frac{k^2}{3}x^2} \right) \quad (21)$$

and

$$H = \frac{k}{3}x^2p - 2\sqrt{\frac{-p}{k}}, \quad (22)$$

where

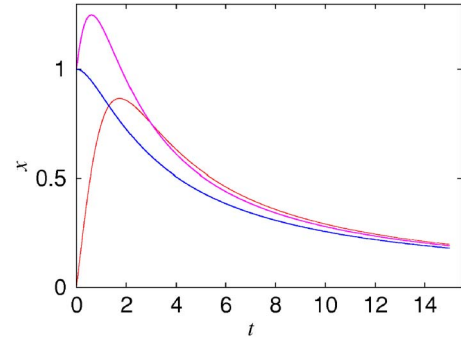


FIG. 4. (Color online) Solution plot for case  $\lambda_1 = 0$  for different initial conditions.

$$p = - \frac{k}{\left( k\dot{x} + \frac{k^2}{3}x^2 \right)^2}. \quad (23)$$

The above form of Lagrangian can also be deduced from the form (14), by taking the Lagrangian for the limiting case  $\lambda_1 = 0$  as

$$\hat{L} = \lim_{\lambda_1 \rightarrow 0} \frac{2k^2}{27\lambda_1^3} \left( L - \frac{3\lambda_1}{2k}\dot{x} + \frac{9\lambda_1^2}{2k^2} \right).$$

The expressions  $p$  and  $H$  given above can be obtained from this  $\hat{L}$ . The solution plot [corresponding to solution (3)] is given in Fig. 4.

One can also calculate the flow function in the case  $\lambda_1 \leq 0$ . Again from Eq. (18) we find that the divergence of the flow function for  $\lambda_1 \leq 0$  is  $\Lambda = -kx$ . But from the general expression for the solution, one can always choose  $x(t) > 0$  for all finite  $t$ . So one might conclude that the system is dissipative (because  $\Lambda < 0$ ) inspite of its just proved conservative nature. In order to overcome this incompatibility, and noting that for the case  $\lambda_1 < 0$ , the dynamical variable  $x(t)$  asymptotically tends to a constant value  $x(\infty) = 3\sqrt{|\lambda_1|}/k$ , we again define the condition for conservative flow is that the time average of the divergence of the flow function with reference to the asymptotic ( $t \rightarrow \infty$ ) value should vanish:

$$\bar{\Lambda} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [\Lambda(t) - \Lambda(\infty)] dt = 0. \quad (24)$$

In the present cases, we have

$$\bar{\Lambda} = - \lim_{T \rightarrow \infty} \frac{k}{T} \int_0^T [x(t) - x(\infty)] dt. \quad (25)$$

For the case  $\lambda_1 = 0$ , from the solution (3) we note that  $x(\infty) = 0$  and  $\int_0^T \Lambda(t) dt = -3 \ln[1 + kT(2I_1 + T)/6I_2]$  so that  $\bar{\Lambda} = 0$  from Eq. (25). On the other hand for  $\lambda_1 < 0$ ,  $\int_0^T \Lambda(t) dt = -3\sqrt{|\lambda_1|}T - 3 \ln[(I_1 + I_1 I_2 e^{-T\sqrt{|\lambda_1|}} + e^{-2T\sqrt{|\lambda_1|}})/(1 + I_1 + I_1 I_2)]$  and  $\Lambda(\infty) = -3\sqrt{|\lambda_1|}$  so that  $\bar{\Lambda}$  also vanishes here.

We note that when the variable dependent dissipative term  $kx\dot{x}$  is absent in Eq. (2) the resultant system becomes an undamped, conservative anharmonic oscillator, that is,  $\ddot{x} + (k^2/9)x^3 + \lambda_1 x = 0$  or equivalently  $\dot{x} = y, \dot{y} = -(k^2/9)x^3 - \lambda_1 x$ .



The divergence of the flow function  $\Lambda$  now vanishes trivially in this case which is in conformity with the Hamiltonian nature of the anharmonic oscillator. On the other hand, if we add a constant damping alone then the equation of motion reads as  $\ddot{x} + k\dot{x} + (k^2/9)x^3 + \lambda_1 x = 0$ , or equivalently  $\dot{x} = y$ ,  $\dot{y} = -ky - (k^2/9)x^3 - \lambda_1 x$ , the divergence of the flow function ( $\Lambda = -k$ ) becomes a negative constant for all values of  $t$  which clearly confirms that the system is a dissipative one. However, the variable dependent dissipative term  $kx\dot{x}$  in Eq. (2), or equivalently Eq. (18), makes the divergence of the flow function  $\Lambda = -kx$ , to be a nonconstant time varying function. In this case, only the time averaged flow function (19) vanishes, as noted above and in Sec. IV B, and the system (2) exhibits both conservative as well as dissipativelike structures depending on the sign of the parameter  $\lambda_1$ , even while admitting Lagrangian and Hamiltonian descriptions in both cases.

## VI. CONNECTION WITH REVERSIBLE DYNAMICAL SYSTEMS

Interestingly, we also note that the system (2) is an example of a reversible dynamical system: Under the transformation

$$S:(x, y, t) \mapsto (-x, y, -t), \quad (26)$$

the equivalent first order system (18) remains invariant. In the literature, reversible dynamical systems have been shown to play an important role within the Hamiltonian as well as non-Hamiltonian dynamics, see, for example, Refs. [8–19]. In this direction, Politi *et al.* [9] have pointed out the possibility of coexistence of both conservative and dissipative behaviors in non-Hamiltonian systems depending upon the value of a control parameter. In particular they have shown that for critical values of the external field the model which they considered exhibits symmetry-breaking bifurcation and the phase space structure changes from conservative to dissipative either in a continuous or discontinuous manner.

The dynamical system studied here clearly demonstrates with explicit solutions how flows with nonconstant time varying divergence can exhibit both conservative type ( $\lambda_1 > 0$ ) and dissipative/frontlike ( $\lambda_1 \leq 0$ ) structures. This is also evident from local analysis of system (18). For example, in the case  $\lambda_1 > 0$ , the equilibrium point is (0,0). The corresponding eigenvalues of the Jacobian matrix are  $\pm i\sqrt{|\lambda_1|}$  so that the equilibrium point is of center type. In the case  $\lambda_1 < 0$ , there are three equilibrium points, namely, (0,0),  $(3\sqrt{|\lambda_1|}/k, 0)$  and  $(-3\sqrt{|\lambda_1|}/k, 0)$ . The corresponding eigenvalues of the Jacobian matrix are  $\pm\sqrt{|\lambda_1|}$ ,  $(-\sqrt{|\lambda_1|}, -2\sqrt{|\lambda_1|})$ , and  $(\sqrt{|\lambda_1|}, 2\sqrt{|\lambda_1|})$ , respectively. Consequently, the equilibrium points become saddle, stable node and unstable node, respectively, and give rise to dissipative/frontlike structures as discussed in Sec. V. Finally, in the case  $\lambda_1 = 0$  we have the equilibrium point (0,0). The associated eigenvalues are also (0,0), indicating that the equilibrium is a degenerate one [5] and that  $\lambda_1 = 0$  is the bifurcation value.

As we have shown in the previous Secs. IV and V both conservativelike and dissipativelike structures coexist in the

system (2) depending on the value of the control parameter  $\lambda_1$ . Even then, we have demonstrated explicitly that the system (2) turns out to be a conservative Hamiltonian type for all values of  $\lambda_1$ . In order to make both the above results consistent we have noted that the time averaged divergence of the flow function should vanish.

## VII. LINEARIZING AND CANONICAL TRANSFORMATIONS

At this stage one may also try to trace the reason for the existence of the above type of remarkable oscillatory behavior for Eq. (2). Under the transformation  $x = 3\dot{w}/kw$ , Eq. (2) transforms to a linear third order differential equation

$$\dddot{w} + \lambda_1 \dot{w} = 0, \quad (27)$$

from which also one can trace the solution (13). Further, it gets transformed into a free particle equation  $X'' = 0$ , where prime denotes differentiation with respect to a new variable  $\tau$ , with the point transformation

$$X = \left( \frac{k}{3\lambda_1} - \frac{1}{\sqrt{-\lambda_1}x} \right) e^{-\sqrt{-\lambda_1}t},$$

$$\tau = \left( \frac{k}{3\sqrt{-\lambda_1}} - \frac{1}{x} \right) e^{\sqrt{-\lambda_1}t}. \quad (28)$$

Even then it is very unexpected and unusual to realize such nonlinear oscillator systems having simple properties similar to that of linear oscillators.

In addition to the above, one can also linearize the Eq. (2) through a nonlocal transformation

$$U = xe^{(k/3)\int x(t')dt'}. \quad (29)$$

Under this transformation Eq. (2) gets modified to the form of a linear harmonic oscillator equation

$$\ddot{U} + \lambda_1 U = 0. \quad (30)$$

Note that the above transformation (29) is valid for all the three cases, namely,  $\lambda_1 > 0$ ,  $\lambda_1 = 0$  and  $\lambda_1 < 0$ . For  $\lambda_1 > 0$ , obviously the solution of Eq. (30) is

$$U = A \sin(\omega t + \delta), \quad (31)$$

where  $A$  and  $\delta$  are arbitrary constants and frequency  $\omega = \sqrt{\lambda_1}$ , which is independent of the amplitude. Consequently, from the forms of the general solutions (31) and (13) we can identify the obvious canonical transformation

$$x = \frac{U}{1 - \frac{k}{3\lambda_1}P}, \quad (32)$$

where  $P = \dot{U}$ , so that using Eq. (15) for the canonically conjugate momentum  $p$ , we have

$$p = P \left( 1 - \frac{k}{6\lambda_1}P \right). \quad (33)$$

It is straightforward to check that when  $U$  and  $P$  are canonical so do  $x$  and  $p$  (and vice versa) and that the Hamiltonian  $H$

in Eq. (16) can be rewritten as the standard linear harmonic oscillator Hamiltonian

$$H = \frac{1}{2}(P^2 + \lambda_1 U^2). \quad (34)$$

More interestingly, even for  $\lambda_1 < 0$ , making use of similar argument for the aperiodic bounded solution (20), we can write the canonical transformation

$$x = \frac{U}{1 + \frac{k}{3\lambda_1}P}, \quad p = P \left( 1 + \frac{k}{6\lambda_1}P \right), \quad (35)$$

where  $P = \dot{U}$ , so that the Hamiltonian (16) (for  $\lambda_1 < 0$ ) is mapped on to the unbounded (wrong sign) linear harmonic oscillator

$$H = \frac{1}{2}(P^2 - \omega^2 U^2), \quad \omega = \sqrt{|\lambda_1|}. \quad (36)$$

Finally for the  $\lambda_1 = 0$  case, we can identify the canonical transformation

$$x = \frac{3P}{kU}, \quad p = -\frac{kU^2}{6}, \quad (37)$$

so as to transform to a freely falling particle Hamiltonian  $H = P^2/2 - \sqrt{\frac{2}{3}}U$ . However, we suspect that there may exist a canonical transformation for this specific case  $\lambda_1 = 0$ , which will take the corresponding form of Eq. (2) to a free particle, though we could not succeed to find it so far.

Of course, it is well known that linearization in general does not ensure preservation of properties of linear systems in the nonlinear case. The classical example is the Cole-Hopf transformation through which the nonlinear Burgers equation is transformed into the linear heat equation [5]. This is also true in the case of various soliton equations such as Korteweg–de Vries, sine-Gorden, nonlinear Schrödinger, and other equations which are all linearizable in certain sense. Nevertheless in the present example of Eq. (2), the amplitude independence of frequency does indeed get preserved. These considerations also raise the question of identifying the most general class of nonlinear dynamical systems of the form  $\ddot{x} = f(x, \dot{x}, t)$  that has solutions whose frequency is independent of the amplitude as in the case of the linear harmonic oscillator, a point which is under study presently.

## VIII. GENERALIZATION AND OTHER EXAMPLES

### A. Generalization

We also observed that the above exceptional properties admitted by Eq. (2) is also common for a class of nonlinear systems. In particular we find that the modified Emden-type equation with linear term and constant external forcing

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1 k_2}{3}x^2 + \lambda_1 x + \lambda_2 = 0, \quad (38)$$

where  $k_1$ ,  $k_2$ ,  $\lambda_1$  and  $\lambda_2$  are parameters, for which also one can obtain similar properties as discussed above for Eq. (2)

when the parameters satisfy the specific condition  $\lambda_1 = 2k_2^2/9 + k_1\lambda_2/k_2$ . To be specific, for the case  $k_1\lambda_2/k_2 - k_2^2/9 > 0$ , the explicit periodic solution takes the form

$$\bar{x}(t) = \frac{A \sin(\bar{\omega}t + \delta)}{1 - \left(\frac{k}{3\bar{\omega}}\right)A \cos(\bar{\omega}t + \delta)}, \quad (39)$$

$$0 \leq A < \frac{3\bar{\omega}}{k}, \quad \bar{\omega} = \sqrt{\lambda_1 - \frac{k_2^2}{9}},$$

where  $\bar{x}(t) = x(t) + k_2/k_1$ . For  $k_1\lambda_2/k_2 - k_2^2/9 < 0$ , the system becomes aperiodic.

### B. Comparison with other nonlinear oscillators

Finally, it is of interest to compare the dynamics of Eq. (2) with another interesting nonlinear oscillator [5,26] of the form

$$\ddot{x} - \frac{\lambda x \dot{x}^2}{1 + \lambda x^2} + \frac{\omega_0^2 x}{1 + \lambda x^2} = 0, \quad (40)$$

which possesses exact periodic solution  $x(t) = A \sin(\Omega t + \delta)$ , where the frequency  $\Omega = \omega_0 / \sqrt{1 + \lambda A^2}$ , exhibiting the characteristic amplitude-dependent frequency of nonlinear oscillators in spite of the sinusoidal nature of the solution Eq. (40). Equation (40) has a natural generalization in three dimensions [27] and these systems can be also quantized exhibiting many interesting features and can be interpreted as an oscillator constrained to move on a three sphere [28]. Several generalizations of these systems to  $N$  degrees of freedom are also possible [29]. But all these nice properties are subject to the fact that the frequency is amplitude dependent. It is only Eq. (2) which admits the amplitude independence property of frequency for a nonlinear oscillator. Studies of such systems will have important implications in developing nonlinear systems whose frequency remain unchanged even with the addition of suitable nonlinearity. Also, Eq. (40) when written as a system of two first order equations of the form (18), with

$$f_1(x, y) = y \quad \text{and} \quad f_2(x, y) = \frac{\lambda x y^2}{1 + \lambda x^2} - \frac{\omega_0^2 x}{1 + \lambda x^2},$$

the divergence of the flow function  $\Lambda = 2\lambda xy / (1 + \lambda x^2) \neq 0$ . On the other hand Eq. (40) is also a Hamiltonian system [26] with the Hamiltonian given by

$$H = \frac{1}{2} \left[ p^2 (1 + \lambda x^2) + \frac{\omega_0^2 x}{1 + \lambda x^2} \right],$$

even though  $\Lambda \neq 0$ . Once again it is only

$$\bar{\Lambda} = \frac{1}{T} \int_0^T \frac{2\lambda xy}{1 + \lambda x^2} dt = 0.$$

This again confirms that for a conservative system it is necessary that the average of the divergence of the flow function vanishes.

## IX. CONCLUSION

In this work, we have identified a nonlinear oscillator which exhibits unusual oscillatory properties: Purely harmonic oscillation whose frequency is completely independent of the amplitude and is the same as that of the linear harmonic oscillator. The conservative Hamiltonian nature of the system has been established, which also necessitates the generalization of the definition of conservative systems in terms of the divergence of the flow function. Further, even for an apparently dissipative type/aperiodic (frontlike) type system, the Hamiltonian structure is preserved and the definition of the conservative system in terms of the divergence of flow function needs to be generalized. In addition to pointing out that the system (2) is an example of reversible dynamical system we have demonstrated that both conservative and dissipativelike behaviors can coexist in conservative Hamiltonian systems of the type considered here depending on the value of the control parameter. We believe identification of such nonlinear systems having the basic property of linear systems will have considerable practical application as the effect of higher harmonics is completely suppressed. Also it is of interest to consider the quantum mechanical version of the system (16) and its higher dimensional generalizations as well as the effect of external forcing and additional damping, which are being pursued.

We have also shown that Eq. (2) admits certain interesting geometrical properties. For example, under the invertible point transformation (28), Eq. (2) gets transformed to the free particle equation whereas with the appropriate choice of canonical transformations one can transform it to a simple harmonic oscillator (or to an unbounded oscillator or to a freely falling particle depending on the value of the parameter  $\lambda_1$ ) equation. Interestingly, the very same nonlinear oscillator can also be transformed to a third order linear equation through a nonlocal transformation, while yet another nonlocal transformation transforms it to a linear harmonic oscillator. As far as our knowledge goes no such single nonlinear oscillator possesses such large class of transformation properties. We believe that exploring such nonlinear equations will be highly rewarding in understanding nonlinear systems and further work is in progress in this direction.

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